

Spectral distribution method for neutrinoless double-beta decay nuclear transition matrix elements: Binary correlation results

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Abstract

Neutrinoless double-beta decay nuclear transition matrix elements are generated by an effective two-body transition operator and it consists of Gamow-Teller like and Fermi like (also tensor) operators. Spectral distribution method for the corresponding transition strengths (squares of the transition matrix elements) involves convolution of the transition strength density generated by the non-interacting particle part of the Hamiltonian with a spreading function generated by the two-body part of the Hamiltonian. Extending the binary correlation theory for spinless embedded k -body ensembles to ensembles with proton-neutron degrees of freedom, we establish that the spreading function is a bivariate Gaussian for transition operators $\mathcal{O}(k_O)$ that change k_O number of neutrons to k_O number of protons. Towards this end, we have derived the formulas for the fourth-order cumulants of the spreading function and calculated their values for some heavy nuclei; they are found to vary from ~ -0.4 to -0.1 . Also for nuclei from ^{76}Ge to ^{238}U , the bivariate correlation coefficient is found to vary from $\sim 0.6 - 0.8$ and these values can be used as a starting point for calculating nuclear transition matrix elements using the spectral distribution method.

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I. INTRODUCTION

Double- β decay (DBD) is an extremely rare weak-interaction process in which two identical nucleons inside the nucleus undergo decay with or without emission of neutrinos. Theoretically, the two neutrino double beta decay ($2\nu\beta^-\beta^-$) process was first predicted by Mayer [1] following the suggestion of Wigner. This process has been observed in more than 10 nuclei and best adopted values for the half-lives were tabulated recently by Barabash [2]. In 1937, following the suggestion of E. Majorana [3], Racah [4] pointed out the possibility of neutrinoless double- β decay (NDBD or $0\nu\beta^-\beta^-$). Furry [5] in 1939, for the first time calculated NDBD half-lives. Fundamental significance of NDBD is that its experimental confirmation will tell us about lepton number violation in nature and that neutrino is a Majorana particle. More importantly, NDBD gives a value or a bound on neutrino mass [6] provided the half-lives are known experimentally and the corresponding nuclear transition matrix elements (NTME) are obtained using a reliable nuclear model. So far only Klapdor et al [7] claimed to have evidence (at a confidence level of 4.2σ) for $0\nu\beta^-\beta^-$ in ^{76}Ge . At present large number of NDBD experiments are being carried out and many others are in development stage in various laboratories around the world. The nuclei being considered are ^{48}Ca , ^{76}Ge , ^{82}Se , ^{100}Mo , ^{116}Cd , ^{130}Te , ^{136}Xe , ^{150}Nd and so on [6]. Following this, several nuclear models are employed for calculating the NTME for various candidate nuclei mentioned above. Some of the models used for NDBD studies are shell model using recent state-of-the-art large scale calculations [8, 9], quasi-particle random-phase approximation with various extensions [10, 11], interacting boson model [12], pseudo-SU(3) model [13], projected Hartree-Fock-Bogoliubov method including pairing plus quadrupole-quadrupole interaction [14, 15], generating coordinate method with particle number and angular momentum projection [16].

Statistical spectral distribution theory [17] gives a method for calculating transition strengths (squares of transition matrix elements) generated by a transition operator. This theory starts with shell model spectroscopic spaces and the same shell model inputs (single particle energies and effective two-body interactions). Here one constructs smoothed forms (spectral distributions) for various observables ignoring the fluctuations and this is based on random matrix representation of Hamiltonians (also other operators), unitary decompositions of the operators and quantum chaos. Spectral distribution theory has been applied in the past, with various approximations, to a variety of problems in nuclear structure and they include (i) bound on time-reversal non-invariant part of the nucleon-nucleon interaction [18], (ii) single particle transfer [19], (iii) β -decay rates for

pre-supernovae evolution [20, 21], (iv) giant dipole widths [22], and (v) parity breaking matrix elements in compound nuclei [23]. Our purpose in this paper is to describe spectral distribution method for the NTME for NDBD and establish that the spreading function that enters in the theory is close to a bivariate Gaussian. With this result, it is possible to apply in future spectral distribution method to NDBD. Now we will give a preview.

In Sec. II, we describe briefly the structure of the $0\nu\beta^-\beta^-$ transition operator which is a two-body operator. Section III describes the spectral distribution method for NDBD. Section IV gives the binary correlation theory for traces of operators over two-orbit (proton and neutron) configurations. Using these, in Sec. V, derived are the results for the fourth-order bivariate cumulants and the bivariate correlation coefficient for the spreading transition strength density function appropriate for NDBD. Numerical results are presented for the fourth-order cumulants to demonstrate that the transition strength density is close to a bivariate Gaussian. In addition, for several heavy nuclei, the values for the bivariate correlation coefficient are presented. Finally, Sec. VI gives conclusions.

II. 0ν DOUBLE-BETA DECAY TRANSITION OPERATOR

Half-life for 0ν double-beta decay, for the 0_i^+ ground state (gs) of a initial even-even nucleus decay to the 0_f^+ gs of the final even-even nucleus is given by [24]

$$[T_{1/2}^{0\nu}(0_i^+ \rightarrow 0_f^+)]^{-1} = G^{0\nu} |M^{0\nu}(0^+)|^2 \left(\frac{\langle m_\nu \rangle}{m_e} \right)^2, \quad (1)$$

where $\langle m_\nu \rangle$ is the effective neutrino mass (a combination of neutrino mass eigenvalues and it also involves neutrino mixing matrix). The $G^{0\nu}$ is phase space integral (kinematical factor) dependent on charge, mass and available energy for NDBD process; tabulations for $G^{0\nu}$ are available [25, 26]. The $M^{0\nu}$ is the NTME generated by the NDBD transition operator and it is a sum of a Gamow-Teller like (M_{GT}), Fermi like (M_F) and tensor (M_T) two-body operators. As it is well known that the tensor part contributes only up to 10% of the matrix elements [27, 28], we will neglect the tensor part. Then we have, from the closure approximation which is well justified for NDBD [24],

$$M^{0\nu}(0^+) = M_{GT}^{0\nu}(0^+) - \frac{g_V^2}{g_A^2} M_F^{0\nu}(0^+) = \langle 0_f^+ || \mathcal{O}(2 : 0\nu) || 0_i^+ \rangle, \quad (2)$$

$$\mathcal{O}(2 : 0\nu) = \sum_{a,b} \mathcal{H}(r_{ab}, \bar{E}) \tau_a^+ \tau_b^+ \left(\sigma_a \cdot \sigma_b - \frac{g_V^2}{g_A^2} \right).$$

As seen from Eq. (2), NDBD half-lives are generated by the two-body transition operator $\mathcal{O}(2 : 0\nu)$; note that a, b label nucleons. The g_A and g_V are the weak axial-vector and vector coupling constants. The $\mathcal{H}(r_{ab}, \bar{E})$ in Eq. (2) is called ‘neutrino potential’. Here \bar{E} is the average energy of the virtual intermediate states used in the closure approximation. The form given by Eq. (2) is justified *only if the exchange of the light Majorana neutrino is indeed the mechanism responsible for the NDBD*. The neutrino potential, defining completely the two-body NDBD transition operator $\mathcal{O}(2 : 0\nu)$ is, to a good approximation, given by [27–31],

$$\mathcal{H}(r_{ab}, \bar{E}) = \frac{R}{r_{ab}} \Phi(r_{ab}, \bar{E}) . \quad (3)$$

Here, R in fm units is the nuclear radius and similarly r_{ab} is in fm units. The function Φ is given by [27, 28, 31],

$$\Phi(r_{ab}, \bar{E}) = \frac{2}{\pi} \left[\sin \left(\frac{\bar{E} r_{ab}}{\hbar c} \right) f_1 \left(\frac{\bar{E} r_{ab}}{\hbar c} \right) - \cos \left(\frac{\bar{E} r_{ab}}{\hbar c} \right) f_2 \left(\frac{\bar{E} r_{ab}}{\hbar c} \right) \right] . \quad (4)$$

In Eq. (4), $f_1(x) = -\int_x^\infty t^{-1} \cos t \, dt = Ci(x) = \gamma + \ln x + \int_0^x t^{-1} (\cos t - 1) \, dt$ and $f_2(x) = -\int_x^\infty t^{-1} \sin t \, dt = Si(x) - \frac{\pi}{2}$; $Si(x)$ and $Ci(x)$ are the sine and cosine integrals. It is useful to mention that $\Phi(r_{ab}, \bar{E}) \sim \exp(-\frac{3}{2} \frac{\bar{E}}{\hbar c} r_{ab})$. Note that $\hbar c = 197.327 \text{ MeV fm}$. The effects of short-range correlations in the wavefunctions are usually taken into account by multiplying the wavefunction by the Jastrow function $[1 - \gamma_3 e^{-\gamma_1 r_{ab}^2} (1 - \gamma_2 r_{ab}^2)]$ [28, 29, 31]. Now keeping the wavefunctions unaltered, the Jastrow function can be incorporated into $\mathcal{H}(r_{ab}, \bar{E})$ giving an effective $\mathcal{H}_{eff}(r_{ab}, \bar{E})$,

$$\mathcal{H}(r_{ab}, \bar{E}) \rightarrow \mathcal{H}_{eff}(r_{ab}, \bar{E}) = \mathcal{H}(r_{ab}, \bar{E}) [1 - \gamma_3 e^{-\gamma_1 r_{ab}^2} (1 - \gamma_2 r_{ab}^2)]^2 . \quad (5)$$

The standard set of values for the parameters γ_1 , γ_2 and γ_3 are given ahead. The most important point about Eq. (1), as mentioned earlier, is that an experimental value of (bound on) $T_{1/2}^{0\nu}$ will give a value for (bound on) neutrino mass via Eq. (1) provided we know the value of the square of NTME $M^{0\nu}(0^+)$ of the NDBD two-body transition operator $\mathcal{O}(2 : 0\nu)$, connecting the ground state of the initial and final even-even nuclei involved.

Let us say that for the nuclei under consideration, protons are in the single particle (sp) orbits j^p and similarly neutrons in j^n . Using the usual assumption that the radial part of the sp states are those of the harmonic oscillator, the proton sp states are completely specified by $(\mathbf{n}^p, \ell^p, j^p)$ with \mathbf{n}^p denoting oscillator radial quantum number so that for a oscillator shell \mathcal{N}^p , $2\mathbf{n}^p + \ell^p = \mathcal{N}^p$. Similarly the neutron sp states are $(\mathbf{n}^n, \ell^n, j^n)$. In terms of the creation (a^\dagger) and annihilation (a) operators, normalized two-particle (antisymmetrized) creation operator $A_\mu^J(j_1 j_2) =$

$(1 + \delta_{j_1 j_2})^{-1/2} (a_{j_1}^\dagger a_{j_2}^\dagger)_\mu^J$ and then $A_\mu^J |0\rangle = |(j_1 j_2) J \mu\rangle$ is a normalized two-particle state. With these the NDBD transition operator can be written as,

$$\mathcal{O}(2 : 0\nu) = \sum_{j_1^p \geq j_2^p : j_3^n \geq j_4^n ; J} \mathcal{O}_{j_1^p j_2^p : j_3^n j_4^n}^J(0\nu) \sum_{\mu} A_\mu^J(j_1^p j_2^p) \{A_\mu^J(j_3^n j_4^n)\}^\dagger. \quad (6)$$

Note that $\mathcal{O}_{j_1^p j_2^p : j_3^n j_4^n}^J(0\nu) = \langle (j_1^p j_2^p) JM | \mathcal{O}(2 : 0\nu) | (j_3^n j_4^n) JM \rangle_a$ are two-body matrix elements (TBME) and ‘a’ denotes that we are considering antisymmetrized two-particle wavefunctions; J is even for $j_1 = j_2$ or $j_3 = j_4$. Numerical values for the TBME’s in Eq. (6) follow from the definition of $\mathcal{O}(2 : 0\nu)$ in Eq. (2) which can be expressed as

$$\mathcal{O}(2 : 0\nu) = \sum_{a,b} \tau_a^+ \tau_b^+ (\mathcal{O}^{GT} - (g_V^2/g_A^2) \mathcal{O}^F); \quad (7)$$

$$\mathcal{O}^{GT} = \sigma_a \cdot \sigma_b \mathcal{H}_{eff}(r_{ab}, \overline{E}), \quad \mathcal{O}^F = \mathcal{H}_{eff}(r_{ab}, \overline{E}).$$

Note that $\tau_a^+ \tau_b^+$ simply changes two neutrons to two protons and for the remaining parts, to obtain the TBME, we use Brody-Moshinsky brackets [32–34]; see [27] for an alternative approach. Then, the TBME are given by,

$$\begin{aligned} \langle (j_1^p j_2^p) JM | \mathcal{O}(2 : 0\nu) | (j_3^n j_4^n) JM \rangle_a &= \frac{1}{\sqrt{(1 + \delta_{j_1^p j_2^p})(1 + \delta_{j_3^n j_4^n})}} \\ &\times \sum_{L,S} \left[2S(S+1) - 3 - \frac{g_V^2}{g_A^2} \right] \chi \left\{ \begin{matrix} \ell_1^p & \ell_2^p & L \\ \frac{1}{2} & \frac{1}{2} & S \\ j_1^p & j_2^p & J \end{matrix} \right\} \chi \left\{ \begin{matrix} \ell_3^n & \ell_4^n & L \\ \frac{1}{2} & \frac{1}{2} & S \\ j_3^n & j_4^n & J \end{matrix} \right\} \\ &\times \sum_{n,\ell;N,L'} [1 + (-1)^{\ell+S}] \langle n\ell, NL', L | \mathbf{n}_1^p \ell_1^p, \mathbf{n}_2^p \ell_2^p, L \rangle \times \langle n'\ell, NL', L | \mathbf{n}_3^n \ell_3^n, \mathbf{n}_4^n \ell_4^n, L \rangle \\ &\times \sum_p B(n\ell, n'\ell, p) I_p. \end{aligned} \quad (8)$$

Here, $\chi\{---\}$ are the $9j$ -coefficients, $\langle \cdots | \cdots \rangle$ are Brody-Moshinsky brackets [32–34], $B(\cdots)$ are Brody, Jacob and Moshinsky coefficients [33] and I_p are Talmi integrals [34]. It is important to mention that antisymmetrized matrix elements for $\mathcal{O}(2 : 0\nu)$ are used in the shell model calculations of NTME while in QRPA related studies non-antisymmetrized matrix elements are employed [28, 35].

There are a number of parameters in the NDBD transition operator and some of them are usually varied in the calculations. The parameters are: (i) $R = 1.1A^{1/3} - 1.2A^{1/3}$ fm [31]; (ii) $b = 1.003A^{1/6}$ fm [27]; (iii) $\overline{E} = 1.12A^{1/2}$ MeV [30]; (iv) $g_A/g_V = 1$ (quenched) or 1.25 (unquenched) [31]; (v) three different choices for the parameters $(\gamma_1, \gamma_2, \gamma_3)$ in Eq. (5) are (1.1, 0.68, 1) [Miller-Spencer], (1.52, 1.88, 0.46) [CD-Bonn] and (1.59, 1.45, 0.92) [AV18]; these values are taken from [28, 31]. It is useful to mention that the kinematical factor $G^{0\nu}$ depends on the coupling constant g_A (standard value is 1.25) and also some calculations use different values for r_0 in $R = r_0A^{1/3}$ fm; the standard value is $r_0 = 1.2$. Then a scaling for $M^{0\nu}$ is [31, 36]

$$(M^{0\nu})' = \left(\frac{g_A}{1.25}\right)^2 \left(\frac{1.2}{r_0}\right) M^{0\nu}. \quad (9)$$

Now we will give the spectral distribution formulation for calculating NTME for NDBD.

III. SPECTRAL DISTRIBUTION METHOD FOR NTME

Following Eqs. (1) and (2) for NDBD half-lives, the corresponding NTME $|M^{0\nu}|^2$ can be viewed as a transition strength (matrix element connecting a given initial state to a final state by a transition operator) generated by the two-body transition operator $\mathcal{O}(2 : 0\nu)$. Therefore, spectral distribution theory [37–41], based on random matrix theory, for transition strength densities, can be applied [42]. Transition strength density is defined as the transition strength multiplied by the state densities at the initial and final energies involved.

Let us consider shell model spherical orbits with angular momenta $j_1^p, j_2^p, \dots, j_r^p$ with m_p protons distributed over them and similarly m_n neutrons over $j_1^n, j_2^n, \dots, j_s^n$ orbits. Then the proton configurations $\widetilde{m}_p = [m_p^1, m_p^2, \dots, m_p^r]$ where m_p^i is number of protons in the orbit j_i^p and $\sum_{i=1}^r m_p^i = m_p$. Similarly the neutron configurations $\widetilde{m}_n = [m_n^1, m_n^2, \dots, m_n^s]$ where m_n^i is number of neutrons in the orbit j_i^n and $\sum_{i=1}^s m_n^i = m_n$. With these, $(\widetilde{m}_p, \widetilde{m}_n)$'s denote proton-neutron configurations. The nuclear effective Hamiltonian is one plus two-body, $H = \mathbf{h} + \mathbf{V}$ and we assume that the one-body part \mathbf{h} includes the mean-field producing part of the two-body part. Thus [43, 44], \mathbf{V} is the irreducible two-body part of H . The state density $I^H(E)$, with $\langle\langle - \rangle\rangle$ denoting trace, can be written as a sum of partial densities defined over $(\widetilde{m}_p, \widetilde{m}_n)$, i.e. $I^{(m_p, m_n)}(E) = \langle\langle \delta(H - E) \rangle\rangle^{(m_p, m_n)} = \sum_{(\widetilde{m}_p, \widetilde{m}_n)} \langle\langle \delta(H - E) \rangle\rangle^{(\widetilde{m}_p, \widetilde{m}_n)} = \sum_{(\widetilde{m}_p, \widetilde{m}_n)} I^{(\widetilde{m}_p, \widetilde{m}_n)}(E) = \sum_{(\widetilde{m}_p, \widetilde{m}_n)} d(\widetilde{m}_p, \widetilde{m}_n) \rho^{(\widetilde{m}_p, \widetilde{m}_n)}(E)$. Here $d(\widetilde{m}_p, \widetilde{m}_n)$ is dimension and $\rho^{(\widetilde{m}_p, \widetilde{m}_n)}(E)$ is the partial density normalized to unity. For nuclear Hamiltonians, it is well under-

stood [17, 40, 45] that the smoothed form for the eigenvalue densities is generated by the action (locally) of EGOE(1+2) [embedded GOE of one plus two-body interactions] operating in the Gaussian domain. This gives,

$$I^{(m_p, m_n)}(E) = \sum_{(\widetilde{m}_p, \widetilde{m}_n)} I_{\mathcal{G}}^{(\widetilde{m}_p, \widetilde{m}_n)}(E) . \quad (10)$$

In Eq. (10), \mathcal{G} denotes Gaussian. The Gaussian partial densities are defined by the centroids $E_c(\widetilde{m}_p, \widetilde{m}_n) = \langle H \rangle^{(\widetilde{m}_p, \widetilde{m}_n)} \sim \langle \mathbf{h} \rangle^{(\widetilde{m}_p, \widetilde{m}_n)}$ and variances $\sigma^2(\widetilde{m}_p, \widetilde{m}_n) = \langle H^2 \rangle^{(\widetilde{m}_p, \widetilde{m}_n)} - [E_c(\widetilde{m}_p, \widetilde{m}_n)]^2 \sim \langle \mathbf{V}^2 \rangle^{(\widetilde{m}_p, \widetilde{m}_n)}$ and, as they are traces over $(\widetilde{m}_p, \widetilde{m}_n)$ spaces, they can be calculated without recourse to H matrix construction. Propagation equations for them, in terms of the sp energies defining \mathbf{h} and TBME defining \mathbf{V} , are easy to write down. Unitary group tensorial structure of \mathbf{h} and \mathbf{V} gives further simplifications of Eq. (10); see [43, 44, 46] for details and applications to fp -shell and also for heavy ($A \gtrsim 150$) nuclear data analysis.

Random matrix theory, based on EGOE(1+2), for the (smoothed) transition strength densities $I_{\mathcal{O}}(E_i, E_f) = I(E_f) |\langle E_f | \mathcal{O} | E_i \rangle|^2 I(E_i)$ allows us to write $I_{\mathcal{O}}$ as a convolution of the corresponding density generated by the mean-field part \mathbf{h} with a spreading bivariate Gaussian $\rho_{biv-\mathcal{G}:\mathcal{O}:\mathbf{V}}$ due to the interaction \mathbf{V} . This gives [37–40], for the square of the \mathcal{O} matrix elements, with $|0_i^+\rangle = |E_i\rangle$ and $|0_f^+\rangle = |E_f\rangle$ where E 's are energies,

$$\begin{aligned} |\langle E_f | \mathcal{O} | E_i \rangle|^2 &= \sum_{(\widetilde{m}_p, \widetilde{m}_n)_i, (\widetilde{m}_p, \widetilde{m}_n)_f} \frac{I_{\mathcal{G}}^{(\widetilde{m}_p, \widetilde{m}_n)_i}(E_i) I_{\mathcal{G}}^{(\widetilde{m}_p, \widetilde{m}_n)_f}(E_f)}{I^{(m_p, m_n)_i}(E_i) I^{(m_p, m_n)_f}(E_f)} \\ &\times |\langle (\widetilde{m}_p, \widetilde{m}_n)_f | \mathcal{O} | (\widetilde{m}_p, \widetilde{m}_n)_i \rangle|^2 \frac{\rho_{biv-\mathcal{G}:\mathcal{O}:\mathbf{V}}^{(\widetilde{m}_p, \widetilde{m}_n)_i, (\widetilde{m}_p, \widetilde{m}_n)_f}(E_i, E_f; E_c^i, E_c^f, \sigma_i, \sigma_f, \zeta)}{\rho_{\mathcal{G}}^{(\widetilde{m}_p, \widetilde{m}_n)_i}(E_i) \rho_{\mathcal{G}}^{(\widetilde{m}_p, \widetilde{m}_n)_f}(E_f)} . \end{aligned} \quad (11)$$

In Eq. (11), $|\langle (\widetilde{m}_p, \widetilde{m}_n)_f | \mathcal{O} | (\widetilde{m}_p, \widetilde{m}_n)_i \rangle|^2$ is the mean square matrix element of \mathcal{O} connecting $(\widetilde{m}_p, \widetilde{m}_n)_i$ and $(\widetilde{m}_p, \widetilde{m}_n)_f$ configurations,

$$\begin{aligned} |\langle (\widetilde{m}_p, \widetilde{m}_n)_f | \mathcal{O} | (\widetilde{m}_p, \widetilde{m}_n)_i \rangle|^2 &= \{d[(\widetilde{m}_p, \widetilde{m}_n)_i] d[(\widetilde{m}_p, \widetilde{m}_n)_f]\}^{-1} \\ &\times \sum_{\alpha \in (\widetilde{m}_p, \widetilde{m}_n)_i, \beta \in (\widetilde{m}_p, \widetilde{m}_n)_f} |\langle (\widetilde{m}_p, \widetilde{m}_n)_f \beta | \mathcal{O} | (\widetilde{m}_p, \widetilde{m}_n)_i \alpha \rangle|^2 . \end{aligned} \quad (12)$$

For later reference it is also useful define,

$$\langle [\mathcal{O}]^\dagger \mathcal{O} \rangle^{(\widetilde{m}_p, \widetilde{m}_n)_i} = d[(\widetilde{m}_p, \widetilde{m}_n)_f] \sum_{(\widetilde{m}_p, \widetilde{m}_n)_f} |\langle (\widetilde{m}_p, \widetilde{m}_n)_f | \mathcal{O} | (\widetilde{m}_p, \widetilde{m}_n)_i \rangle|^2 . \quad (13)$$

Following [37–39], the two centroids E_c^i and E_c^f and the two variances σ_i^2 and σ_f^2 of the marginal densities of the bivariate Gaussian $\rho_{biv-G:\mathcal{O}:\mathbf{V}}$, with some approximations are given by,

$$\begin{aligned} E_c^i &= E_c((\widetilde{m}_p, \widetilde{m}_n)_i) , & E_c^f &= E_c((\widetilde{m}_p, \widetilde{m}_n)_f) , \\ \sigma_i^2 &= \sigma^2((\widetilde{m}_p, \widetilde{m}_n)_i) , & \sigma_f^2 &= \sigma^2((\widetilde{m}_p, \widetilde{m}_n)_f) . \end{aligned} \quad (14)$$

These are the proton-neutron configuration centroids and variances generated by H . Eq. (14) has its basis in Eq. (54) given ahead. Although the general structure and importance of the bivariate correlation coefficient ζ in Eq. (11) is well understood [37–39], an expression for ζ in terms of traces over $(\widetilde{m}_p, \widetilde{m}_n)_i$ and $(\widetilde{m}_p, \widetilde{m}_n)_f$ spaces is not yet available. Then the standard approximation, for completing the theory, is

$$\zeta = \frac{X_{11}[(m_p, m_n)_i]}{\sqrt{X_{20}[(m_p, m_n)_i] X_{02}[(m_p, m_n)_i]}} ; \quad X_{PQ}[(m_p, m_n)_i] = \frac{\langle [\mathcal{O}]^\dagger \mathbf{V}^Q \mathcal{O} \mathbf{V}^P \rangle^{(m_p, m_n)_i}}{\langle [\mathcal{O}]^\dagger \mathcal{O} \rangle^{(m_p, m_n)_i}} . \quad (15)$$

To proceed further, propagation formulas for the traces in Eqs. (12) and (15) are needed. For the trace in Eq. (12), using the results in [50], it is easy to write the propagation formula in terms of the TBME $\mathcal{O}_{--}^{J_0}(0\nu)$ defined by Eq. (6),

$$\begin{aligned} & | \langle (\widetilde{m}_p, \widetilde{m}_n)_f | \mathcal{O} | (\widetilde{m}_p, \widetilde{m}_n)_i \rangle |^2 d[(\widetilde{m}_p, \widetilde{m}_n)_f] \\ &= \sum_{\alpha, \beta, \gamma, \delta} \frac{m_n^i(\alpha) [m_n^i(\beta) - \delta_{\alpha\beta}] [N_p(\gamma) - m_p^i(\gamma)] [N_p(\delta) - m_p^i(\delta) - \delta_{\gamma\delta}]}{N_n(\alpha) [N_n(\beta) - \delta_{\alpha\beta}] N_p(\gamma) [N_p(\delta) - \delta_{\gamma\delta}]} \\ &\times \sum_{J_0} [\mathcal{O}_{\gamma^p \delta^p \alpha^n \beta^n}^{J_0}(0\nu)]^2 (2J_0 + 1) , \end{aligned} \quad (16)$$

$$(\widetilde{m}_p, \widetilde{m}_n)_f = (\widetilde{m}_p, \widetilde{m}_n)_i \times \left(1_{\gamma^p}^+ 1_{\delta^p}^+ 1_{\alpha^n} 1_{\beta^n} \right) .$$

Note that in Eq. (16) the final configuration is defined by removing one neutron from orbit α and another from β and then adding one proton in orbit γ and another in orbit δ . Also, $N_p(\alpha)$ is the degeneracy of the proton orbit α and similarly $N_n(\alpha)$ for the neutron orbit α . In the limit $\rho_{biv}/\rho\rho = 1$ in Eq. (11), substituting the result of Eq. (16) in Eq. (11) gives the NTME for NDBD. However, this zero-th order approximation will not be good as in general it is expected that $\zeta > 0.5$; see Sec. V ahead. Before proceeding to implement the theory given above, it is essential to test the important approximation used in the theory, i.e. the bivariate Gaussian form for

the spreading function generated by \mathbf{V} by calculating the fourth order bivariate cumulants (they will be zero for a bivariate Gaussian). In addition we also need an expression for the bivariate correlation coefficient ζ . To address these two problems and provide generic results, we consider the spreading function defined over proton-neutron spaces i.e.,

$$\rho_{biv:\mathcal{O}:H}^{(m_p, m_n)_i, (m_p, m_n)_f}(E_i, E_f; E_c^i, E_c^f, \sigma_i, \sigma_f, \zeta),$$

where H is a two-body Hamiltonian. We will consider Hamiltonians that preserve (m_p, m_n) and then $H = H_{pp} + H_{nn} + H_{pn}$. This is quite appropriate for heavy nuclei. Adopting the binary correlation theory, the bivariate moments of $\rho_{biv:\mathcal{O}:H}^{(m_p, m_n)_i, (m_p, m_n)_f}$ are evaluated by considering random k -body H operators. Similarly, we represent the transition operator \mathcal{O} by random $k_{\mathcal{O}}$ -body operator that changes $k_{\mathcal{O}}$ number of neutrons to $k_{\mathcal{O}}$ number of neutrons. This is equivalent to using EGOE representation for both H and \mathcal{O} operators [18, 40]. Let us mention that, from now onwards, we consider only the two-orbit configurations (m_p, m_n) [for generality, these are denoted as (m_1, m_2) in the next two sections].

IV. BINARY CORRELATION RESULTS FOR RANDOM HAMILTONIANS

Binary correlation theory for moments defined over a single unitary orbit is given by Mon and French [47, 48] and they correspond to the moments generated by spinless EGOE(k) in the dilute limit (dilute limit is defined in Sec. IV). The theory is extended to certain two-orbit moments by Tomsovic [49]. For the two problems mentioned in Sec. III, we need traces defined over two-orbits (protons and neutrons) with the H preserving the two-orbit symmetry and the transition operator \mathcal{O} acting on a two-orbit configuration generating a unique final two-orbit configuration. In the present section, we will give the basic binary correlation results adopted for this situation and in Sec. V, we will consider their applications.

A. Results for single unitary orbit

Let us begin with a k_H -body operator,

$$H(k_H) = \sum_{\alpha, \beta} v_H^{\alpha\beta} \alpha^\dagger(k_H) \beta(k_H). \quad (17)$$

Here, $\alpha^\dagger(k_H)$ is the k_H particle creation operator and $\beta(k_H)$ is the k_H particle annihilation operator. Similarly, $v_H^{\alpha\beta}$ are matrix elements of the operator H in k_H particle space i.e., $v_H^{\alpha\beta} =$

$\langle k_H \beta \mid H \mid k_H \alpha \rangle$ (it should be noted that Mon and French [47, 48] used operators with daggers to denote annihilation operators and operators without daggers to denote creation operators). Following basic traces will be used throughout,

$$\sum_{\alpha} \alpha^{\dagger}(k) \alpha(k) = \binom{\hat{n}}{k} \Rightarrow \left\langle \sum_{\alpha} \alpha^{\dagger}(k) \alpha(k) \right\rangle^m = \binom{m}{k} . \quad (18)$$

$$\sum_{\alpha} \alpha(k) \alpha^{\dagger}(k) = \binom{N - \hat{n}}{k} \Rightarrow \left\langle \sum_{\alpha} \alpha(k) \alpha^{\dagger}(k) \right\rangle^m = \binom{\tilde{m}}{k} ; \quad \tilde{m} = N - m . \quad (19)$$

$$\begin{aligned} \sum_{\alpha} \alpha^{\dagger}(k) B(k') \alpha(k) &= \binom{\hat{n} - k'}{k} B(k') \\ \Rightarrow \left\langle \sum_{\alpha} \alpha^{\dagger}(k) B(k') \alpha(k) \right\rangle^m &= \binom{m - k'}{k} B(k') . \end{aligned} \quad (20)$$

$$\begin{aligned} \sum_{\alpha} \alpha(k) B(k') \alpha^{\dagger}(k) &= \binom{N - \hat{n} - k'}{k} B(k') \\ \Rightarrow \left\langle \sum_{\alpha} \alpha(k) B(k') \alpha^{\dagger}(k) \right\rangle^m &= \binom{\tilde{m} - k'}{k} B(k') . \end{aligned} \quad (21)$$

Equation (18) follows from the fact that the average should be zero for $m < k$ and one for $m = k$ and similarly, Eq. (19) follows from the same argument except that the particles are replaced by holes. Equation (20) follows first by writing the k' -body operator $B(k')$ in operator form using Eq. (17), i.e.,

$$B(k') = \sum_{\beta, \gamma} v_B^{\beta\gamma} \beta^{\dagger}(k') \gamma(k') , \quad (22)$$

and then applying the commutation relations for the fermion creation and annihilation operators. This gives $\sum_{\beta, \gamma} v_B^{\beta\gamma} \beta^{\dagger}(k') \sum_{\alpha} \alpha^{\dagger}(k) \alpha(k) \gamma(k')$. Now applying Eq. (18) to the sum involving α gives Eq. (20). Eq. (21) follows from the same arguments except one has to assume that $B(k')$ is fully irreducible $\nu = k'$ operator and therefore, it has particle-hole symmetry. For a general $B(k')$ operator, this is valid only in the $N \rightarrow \infty$ limit. Therefore, this equation has to be applied with caution.

Using the definition of the H operator in Eq. (17), we have

$$\begin{aligned}
\overline{\langle H(k_H)H(k_H) \rangle^m} &= \sum_{\alpha, \beta} \overline{\left\{ v_H^{\alpha\beta} \right\}^2} \langle \alpha^\dagger(k_H) \beta(k_H) \beta^\dagger(k_H) \alpha(k_H) \rangle^m \\
&= v_H^2 \left\langle \sum_{\alpha} \alpha^\dagger(k_H) \left\{ \sum_{\beta} \beta(k_H) \beta^\dagger(k_H) \right\} \alpha(k_H) \right\rangle^m \\
&= v_H^2 T(m, N, k_H) .
\end{aligned} \tag{23}$$

Here, H is taken as EGOE(k_H) with all the k_H particle matrix elements being Gaussian variables with zero center and same variance for off-diagonal matrix elements (twice for the diagonal matrix elements). This gives $\overline{(v_H^{\alpha\beta})^2} = v_H^2$ to be independent of α, β labels. It is important to note that in the dilute limit, the diagonal terms [$\alpha = \beta$ in Eq. (23)] in the averages are neglected (as they are smaller by at least one power of $1/N$) and the individual H 's are unitarily irreducible. These assumptions are no longer valid for finite- N systems and hence, evaluation of averages is more complicated. In the dilute limit, we have

$$\begin{aligned}
T(m, N, k_H) &= \left\langle \sum_{\alpha} \alpha^\dagger(k_H) \left\{ \sum_{\beta} \beta(k_H) \beta^\dagger(k_H) \right\} \alpha(k_H) \right\rangle^m \\
&= \binom{\tilde{m} + k_H}{k_H} \left\langle \sum_{\alpha} \alpha^\dagger(k_H) \alpha(k_H) \right\rangle^m \\
&= \binom{\tilde{m} + k_H}{k_H} \binom{m}{k_H} .
\end{aligned} \tag{24}$$

Note that, we have used Eq. (19) to evaluate the summation over β and Eq. (18) to evaluate summation over α in Eq. (24). In the ‘strict’ $N \rightarrow \infty$ limit, we have

$$T(m, N, k_H) \xrightarrow{N \rightarrow \infty} \binom{m}{k_H} \binom{N}{k_H} . \tag{25}$$

In order to incorporate the finite- N corrections, we have to consider the contribution of the diagonal terms. Then, we have,

$$T(m, N, k_H) = \binom{m}{k_H} \left[\binom{\tilde{m} + k_H}{k_H} + 1 \right] . \tag{26}$$

Now we will turn to the fourth order averages.

For averages involving product of four operators of the form $\langle H(k_H)G(k_G)H(k_H)G(k_G) \rangle^m$, with operators H and G independent and of body ranks k_H and k_G respectively, there are two possible ways of evaluating this trace. Either (a) first contract the H operators across the G operator using Eq. (21) and then contract the G operators using Eq. (20), or (b) first contract the G operators across the H operator using Eq. (21) and then contract the H operators using Eq. (20). However, (a) and (b) give the same result only in the ‘strict’ $N \rightarrow \infty$ limit and also for the result incorporating finite N corrections as discussed below. In general, the final result can be expressed as,

$$\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G) \rangle^m} = v_H^2 v_G^2 F(m, N, k_H, k_G) . \quad (27)$$

In the ‘strict’ dilute limit, $F(m, N, k_H, k_G)$ is given by

$$F(m, N, k_H, k_G) = \binom{m - k_H}{k_G} \binom{m}{k_H} \binom{N}{k_H} \binom{N}{k_G} . \quad (28)$$

In order to obtain finite- N corrections to $F(\dots)$, we have to contract over operators whose lower symmetry parts can not be ignored. The operator $H(k_H)$ decomposes into irreducible symmetry parts $\mathcal{F}(s)$ denoted by $s = 0, 1, 2, \dots, k_H$ with respect to the unitary group $SU(N)$. For a k_H -body number conserving operator [48, 50], we have

$$H(k_H) = \sum_{s=0}^{k_H} \binom{m-s}{k_H-s} \mathcal{F}(s) . \quad (29)$$

Here, the $\mathcal{F}(s)$ are orthogonal with respect to m -particle averages, i.e., $\langle \mathcal{F}(s)\mathcal{F}^\dagger(s') \rangle^m = \delta_{ss'}$.

Now, $\langle H(k_H)G(k_G)H(k_H)G(k_G) \rangle^m$ will have four parts,

$$\begin{aligned} & \overline{\langle H(k_H)G(k_G)H(k_H)G(k_G) \rangle^m} \\ &= v_H^2 v_G^2 \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha^\dagger(k_H)\beta(k_H)\gamma^\dagger(k_G)\delta(k_G)\beta^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\gamma(k_G) \rangle^m \\ &+ v_H^2 v_G^2 \sum_{\alpha, \gamma, \delta} \langle \alpha^\dagger(k_H)\alpha(k_H)\gamma^\dagger(k_G)\delta(k_G)\alpha^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\gamma(k_G) \rangle^m \\ &+ v_H^2 v_G^2 \sum_{\alpha, \beta, \gamma} \langle \alpha^\dagger(k_H)\beta(k_H)\gamma^\dagger(k_G)\gamma(k_H)\beta^\dagger(k_H)\alpha(k_H)\gamma^\dagger(k_G)\gamma(k_G) \rangle^m \\ &+ v_H^2 v_G^2 \sum_{\alpha, \delta} \langle \alpha^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\delta(k_G)\alpha^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\delta(k_G) \rangle^m \\ &= X + Y_1 + Y_2 + Z . \end{aligned} \quad (30)$$

Note that we have decomposed each operator into diagonal and off-diagonal parts. We have used the condition that the variance of the diagonal matrix elements is twice that of the off-diagonal matrix elements in the defining spaces to convert the restricted summations into unrestricted summations appropriately to obtain the four terms in the RHS of Eq. (30). Following [47, 49, 51] and applying unitary decomposition to $\gamma\delta^\dagger$ (also $\delta\gamma^\dagger$) in the first two terms and $\alpha\beta^\dagger$ (also $\beta\alpha^\dagger$) in the third term we get X , Y_1 and Y_2 . To make things clear, we will discuss the derivation for X term in detail before proceeding further. Applying unitary decomposition to the operators $\gamma^\dagger(k_G)\delta(k_G)$ and $\gamma(k_G)\delta^\dagger(k_G)$ using Eq. (29), we have

$$X = v_H^2 v_G^2 \sum_{\alpha, \beta, \gamma, \delta} \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \left\langle \alpha^\dagger(k_H)\beta(k_H)\mathcal{F}_{\gamma\delta}^\dagger(s)\beta^\dagger(k_H)\alpha(k_H)\mathcal{F}_{\gamma\delta}(s) \right\rangle^m. \quad (31)$$

Contracting the operators $\beta\beta^\dagger$ across \mathcal{F} 's using Eq. (21) and operators $\alpha^\dagger\alpha$ across \mathcal{F} using Eq. (20) gives,

$$X = v_H^2 v_G^2 \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \binom{\tilde{m} + k_H - s}{k_H} \binom{m-s}{k_H} \sum_{\gamma, \delta} \left\langle \mathcal{F}_{\gamma\delta}^\dagger(s)\mathcal{F}_{\gamma\delta}(s) \right\rangle^m. \quad (32)$$

Inversion of the equation,

$$\sum_{\gamma, \delta} \left\langle \gamma^\dagger(k_G)\delta(k_G)\delta^\dagger(k_G)\gamma(k_G) \right\rangle^m = Q(m) = \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \sum_{\gamma, \delta} \left\langle \mathcal{F}_{\gamma\delta}^\dagger(s)\mathcal{F}_{\gamma\delta}(s) \right\rangle^m, \quad (33)$$

gives,

$$\begin{aligned} \binom{m-s}{k_G-s}^2 \sum_{\gamma, \delta} \left\langle \mathcal{F}_{\gamma\delta}^\dagger(s)\mathcal{F}_{\gamma\delta}(s) \right\rangle^m &= \binom{m-s}{k_G-s}^2 \binom{N-m}{s} \binom{m}{s} [(k_G-s)!s!]^2 \\ &\times (N-2s+1) \sum_{t=0}^s \frac{(-1)^{t-s} [(N-t-k_G)!]^2}{(s-t)!(N-s-t+1)!t!(N-t)!} Q(N-t). \end{aligned} \quad (34)$$

For the average required in Eq. (33), we have

$$Q(m) = \sum_{\gamma, \delta} \left\langle \gamma^\dagger(k_G)\delta(k_G)\delta^\dagger(k_G)\gamma(k_G) \right\rangle^m = \binom{\tilde{m} + k_G}{k_G} \binom{m}{k_G}. \quad (35)$$

Simplifying Eq. (34) using Eq. (35) and using the result in Eq. (32) along with the series summation [47]

$$\sum_{t=0}^s \frac{(-1)^{t-s} (N-t-k_G)! (k_G+t)!}{(s-t)! (t!)^2 (N-s-t+1)!} = \frac{k_G! (N-k_G-s)!}{(N+1-s)!} \binom{k_G}{s} \binom{N+1}{s}, \quad (36)$$

the expression for X is,

$$X = v_H^2 v_G^2 F(m, N, k_H, k_G) ;$$

$$F(m, N, k_H, k_G) = \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \binom{\tilde{m} + k_H - s}{k_H} \binom{m-s}{k_H} \binom{\tilde{m}}{s} \binom{m}{s} \binom{N+1}{s} \quad (37)$$

$$\times \frac{N-2s+1}{N-s+1} \binom{N-s}{k_G}^{-1} \binom{k_G}{s}^{-1} .$$

Although not obvious, X has $k_H \leftrightarrow k_G$ symmetry. This is easy to verify for $k_H, k_G \leq 2$. In the large N limit, Y_1 , Y_2 and Z are neglected as X will make the dominant contribution; see [51] for details on Y_1 , Y_2 and Z . Thus, in all the applications, we use

$$\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G) \rangle^m} = X = v_H^2 v_G^2 F(m, N, k_H, k_G) . \quad (38)$$

An immediate application of these averages is in evaluating the fourth order average $\overline{\langle H^4(k_H) \rangle^m}$. There will be three different correlation patterns that will contribute to this average in the binary correlation approximation (we must correlate in pairs the operators for all moments of order > 2),

$$\begin{aligned} \overline{\langle H^4(k_H) \rangle^m} &= \overline{\langle \underbrace{H(k_H)H(k_H)}_{HH} \underbrace{H(k_H)H(k_H)}_{HH} \rangle^m} \\ &+ \overline{\langle \underbrace{H(k_H)H(k_H)H(k_H)H(k_H)}_{HHHH} \rangle^m} \\ &+ \overline{\langle \underbrace{H(k_H)H(k_H)H(k_H)H(k_H)}_{HHHH} \rangle^m} . \end{aligned} \quad (39)$$

In Eq. (39), we denote the binary correlated pairs of operators with the symbol \underline{HH} . The first two terms on the RHS of Eq. (39) are equal due to cyclic invariance and follow from Eq. (23),

$$\begin{aligned} \overline{\langle \underbrace{H(k_H)H(k_H)}_{HH} \underbrace{H(k_H)H(k_H)}_{HH} \rangle^m} &= \overline{\langle \underbrace{H(k_H)H(k_H)H(k_H)H(k_H)}_{HHHH} \rangle^m} \\ &= \left[\overline{\langle H^2(k_H) \rangle^m} \right]^2 . \end{aligned} \quad (40)$$

Similarly, the third term on the RHS of Eq. (39) follows from Eq. (38),

$$\overline{\langle \underbrace{H(k_H)H(k_H)H(k_H)H(k_H)}_{HHHH} \rangle^m} = v_H^4 F(m, N, k_H, k_H) . \quad (41)$$

Combining Eqs. (39), (40) and (41), $\overline{\langle H^4(k_H) \rangle^m}$ is given by,

$$\overline{\langle H^4(k_H) \rangle^m} = v_H^4 \left[2 \{T(m, N, k_H)\}^2 + F(m, N, k_H, k_H) \right] . \quad (42)$$

B. Results for two unitary orbits

In the NDBD applications (also β decay), fourth order traces over two orbit configurations are needed. Let us consider m particles in two orbits with number of sp states being N_1 and N_2 respectively. Now the m -particle space can be divided into configurations (m_1, m_2) with m_1 particles in the #1 orbit and m_2 particles in the #2 orbit such that $m = m_1 + m_2$. Considering the operator H with fixed body ranks in m_1 and m_2 spaces such that (m_1, m_2) are preserved by this operators, the general form for H is,

$$H(k_H) = \sum_{i+j=k_H, \alpha, \beta, \gamma, \delta} \left[v_H^{\alpha\beta\gamma\delta}(i, j) \right] \alpha_1^\dagger(i) \beta_1(i) \gamma_2^\dagger(j) \delta_2(j) . \quad (43)$$

Now, it is seen that, in the dilute limit,

$$\begin{aligned} & \overline{\langle H^2(k_H) \rangle}^{m_1, m_2} \\ &= \sum_{i+j=k_H} v_H^2(i, j) \sum_{\alpha, \beta, \gamma, \delta} \left\langle \alpha_1^\dagger(i) \beta_1(i) \gamma_2^\dagger(j) \delta_2(j) \beta_1^\dagger(i) \alpha_1(i) \delta_2^\dagger(j) \gamma_2(j) \right\rangle^{m_1, m_2} \\ &= \sum_{i+j=k_H} v_H^2(i, j) \sum_{\alpha, \beta} \left\langle \alpha_1^\dagger(i) \beta_1(i) \beta_1^\dagger(i) \alpha_1(i) \right\rangle^{m_1} \sum_{\gamma, \delta} \left\langle \gamma_2^\dagger(j) \delta_2(j) \delta_2^\dagger(j) \gamma_2(j) \right\rangle^{m_2} \\ &= \sum_{i+j=k_H} v_H^2(i, j) T(m_1, N_1, i) T(m_2, N_2, j) . \end{aligned} \quad (44)$$

Note that $v_H^2(i, j) = \overline{[v_H^{\alpha\beta\gamma\delta}(i, j)]^2}$ and T 's are defined by Eqs. (24) and (25). The ensemble is defined such that $v_H^{\alpha\beta\gamma\delta}(i, j)$ are independent Gaussian random variables with zero center and the variances depend only on the indices i and j . Similarly, with two operators H and G (with body ranks k_H and k_G respectively) that are independent and both preserving (m_1, m_2) , $\overline{\langle H(k_H) G(k_G) H(k_H) G(k_G) \rangle}^{m_1, m_2}$ is given by,

$$\begin{aligned} & \overline{\langle H(k_H) G(k_G) H(k_H) G(k_G) \rangle}^{m_1, m_2} = \\ & \sum_{i+j=k_H, t+u=k_G} v_H^2(i, j) v_G^2(t, u) F(m_1, N_1, i, t) F(m_2, N_2, j, u) , \end{aligned} \quad (45)$$

and therefore,

$$\begin{aligned} \overline{\langle H^4(k_H) \rangle^{m_1, m_2}} &= 2 \left[\sum_{i+j=k_H} v_H^2(i, j) T(m_1, N_1, i) T(m_2, N_2, j) \right]^2 \\ &+ \sum_{i+j=k_H, t+u=k_H} v_H^2(i, j) v_H^2(t, u) F(m_1, N_1, i, t) F(m_2, N_2, j, u) . \end{aligned} \quad (46)$$

Now we apply the formulation given here to derive the formulas for the second and fourth order cumulants defining $\rho_{biv-G; \mathcal{O}: H}^{(m_p, m_n)_i, (m_p, m_n)_f}$.

V. BINARY CORRELATION RESULTS FOR THE BIVARIATE CORRELATION COEFFICIENT AND FOURTH ORDER CUMULANTS FOR THE TRANSITION STRENGTH DENSITY FOR NDBD

A. Transition matrix elements and bivariate strength densities

Our purpose here is to establish that for the $0\nu\beta^-\beta^-$ decay (also for β decay), transition strength densities, locally, are close to bivariate Gaussian form and also to derive a formula for the corresponding bivariate correlation coefficient. With space #1 denoting protons and similarly space #2 neutrons, the general form of the transition operator \mathcal{O} is,

$$\mathcal{O}(k_{\mathcal{O}}) = \sum_{\gamma, \delta} v_{\mathcal{O}}^{\gamma\delta}(k_{\mathcal{O}}) \gamma_1^{\dagger}(k_{\mathcal{O}}) \delta_2(k_{\mathcal{O}}) ; \quad k_{\mathcal{O}} = 2 \text{ for NDBD} . \quad (47)$$

Therefore, in order to derive the form for the transition strength densities generated by \mathcal{O} , it is necessary to deal with two-orbit configurations denoted by (m_1, m_2) , where m_1 is the number of particles in the first orbit (protons) and m_2 in the second orbit (neutrons). Now, the transition strength density $I_{\mathcal{O}}(E_i, E_f)$ is

$$\begin{aligned} I_{\mathcal{O}}^{(m_1, m_2)_i, (m_1, m_2)_f}(E_i, E_f) \\ = I^{(m_1, m_2)_f}(E_f) |\langle (m_1, m_2)_f E_f | \mathcal{O} | (m_1, m_2)_i E_i \rangle|^2 I^{(m_1, m_2)_i}(E_i) , \end{aligned} \quad (48)$$

and the corresponding bivariate moments are

$$\widetilde{M}_{PQ}((m_1, m_2)_i) = \overline{\langle \mathcal{O}^{\dagger}(k_{\mathcal{O}}) H^Q(k_H) \mathcal{O}(k_{\mathcal{O}}) H^P(k_H) \rangle^{(m_1, m_2)_i}} . \quad (49)$$

Note that \widetilde{M} are in general non-central and non-normalized moments. The general form of the operator $H(k_H)$ is given by Eq. (43) and it preserves $(m_1, m_2)_i$'s. However, \mathcal{O} and its hermitian conjugate \mathcal{O}^\dagger do not preserve (m_1, m_2) i.e., $\mathcal{O}(k_{\mathcal{O}}) |m_1, m_2\rangle = |m_1 + k_{\mathcal{O}}, m_2 - k_{\mathcal{O}}\rangle$ and $\mathcal{O}^\dagger(k_{\mathcal{O}}) |m_1, m_2\rangle = |m_1 - k_{\mathcal{O}}, m_2 + k_{\mathcal{O}}\rangle$. Thus, given a $(m_1, m_2)_i$ for an initial state, the $(m_1, m_2)_f$ for the final state generated by the action of \mathcal{O} is uniquely defined and therefore, the bivariate moments defined by Eq. (49) are proper bivariate moments and they are defined by the initial $(m_1, m_2)_i$. For completeness, let us mention that given the marginal centroids (ϵ_i, ϵ_f) , widths (σ_i, σ_f) and the bivariate correlation coefficient ζ_{biv} , the normalized bivariate Gaussian is defined by,

$$\begin{aligned} \rho_{\text{biv-}\mathcal{G};\mathcal{O}}(E_i, E_f) &= \rho_{\text{biv-}\mathcal{G};\mathcal{O}}(E_i, E_f; \epsilon_i, \epsilon_f, \sigma_i, \sigma_f, \zeta_{biv}) \\ &= \frac{1}{2\pi\sigma_i\sigma_f\sqrt{(1-\zeta_{biv}^2)}} \\ &\times \exp -\frac{1}{2(1-\zeta_{biv}^2)} \left\{ \left(\frac{E_i - \epsilon_i}{\sigma_i} \right)^2 - 2\zeta_{biv} \left(\frac{E_i - \epsilon_i}{\sigma_i} \right) \left(\frac{E_f - \epsilon_f}{\sigma_f} \right) + \left(\frac{E_f - \epsilon_f}{\sigma_f} \right)^2 \right\}. \end{aligned} \quad (50)$$

B. Formulas for the bivariate moments

Using binary correlation approximation, we derive formulas for the first four moments $\widetilde{M}_{PQ}((m_1, m_2)_i)$, $P+Q \leq 4$ of $I_{\mathcal{O}}^{(m_1, m_2)_i, (m_1, m_2)_f}(E_i, E_f)$ for any $k_{\mathcal{O}}$ by representing $H(k_H)$ and $\mathcal{O}(k_{\mathcal{O}})$ operators by independent EGOEs and assuming $H(k_H)$ is a k_H -body operator preserving (m_1, m_2) 's. Note that the ensemble averaged k_H -particle matrix elements of $H(k_H)$ are $v_H^2(i, j)$ with $i + j = k_H$ [see Eq. (43)] and similarly the ensemble average of $(v_{\mathcal{O}}^\delta)^2$ is $v_{\mathcal{O}}^2$. From now on, we use $(m_1, m_2)_i = (m_1, m_2)$. Using Eq. (47) and applying the basic rules given by Eqs. (18) and (19), we have

$$\begin{aligned} \widetilde{M}_{00}(m_1, m_2) &= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \mathcal{O}(k_{\mathcal{O}}) \rangle}^{m_1, m_2} \\ &= \sum_{\gamma, \delta} \overline{\left\{ v_{\mathcal{O}}^{\gamma\delta} \right\}^2} \left\langle \delta_2^\dagger(k_{\mathcal{O}}) \gamma_1(k_{\mathcal{O}}) \gamma_1^\dagger(k_{\mathcal{O}}) \delta_2(k_{\mathcal{O}}) \right\rangle^{m_1, m_2} \\ &= v_{\mathcal{O}}^2 \begin{pmatrix} \widetilde{m}_1 \\ k_{\mathcal{O}} \end{pmatrix} \begin{pmatrix} m_2 \\ k_{\mathcal{O}} \end{pmatrix}. \end{aligned} \quad (51)$$

Trivially, $\widetilde{M}_{10}(m_1, m_2)$ and $\widetilde{M}_{01}(m_1, m_2)$ will be zero as $H(k_H)$ is represented by EGOE(k_H). Thus, $\widetilde{M}_{PQ}(m_1, m_2)$ are central moments. Moreover, by definition, all the odd-order moments, i.e., $\widetilde{M}_{PQ}(m_1, m_2)$ with $\text{mod}(P+Q, 2) \neq 0$, will be zero. Now, the \widetilde{M}_{11} is given by,

$$\begin{aligned} \widetilde{M}_{11}(m_1, m_2) &= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}})H(k_H)\mathcal{O}(k_{\mathcal{O}})H(k_H) \rangle^{m_1, m_2}} \\ &= v_{\mathcal{O}}^2 \sum_{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \delta_2; i+j=k_H} v_H^2(i, j) \left\langle \gamma_1^\dagger(k_{\mathcal{O}})\alpha_1(i)\beta_1^\dagger(i)\gamma_1(k_{\mathcal{O}})\beta_1(i)\alpha_1^\dagger(i) \right\rangle^{m_1} \\ &\quad \times \left\langle \delta_2(k_{\mathcal{O}})\alpha_2(j)\beta_2^\dagger(j)\delta_2^\dagger(k_{\mathcal{O}})\beta_2(j)\alpha_2^\dagger(j) \right\rangle^{m_2}. \end{aligned} \quad (52)$$

Then, contracting over the $\gamma^\dagger\gamma$ and $\delta\delta^\dagger$ operators, respectively in the first and second traces in Eq. (52) using Eqs. (20) and (21) appropriately, we have

$$\begin{aligned} \widetilde{M}_{11}(m_1, m_2) &= v_{\mathcal{O}}^2 \sum_{i+j=k_H} v_H^2(i, j) \binom{\widetilde{m}_1 - i}{k_{\mathcal{O}}} \binom{m_2 - j}{k_{\mathcal{O}}} \\ &\quad \times T(m_1, N_1, i) T(m_2, N_2, j). \end{aligned} \quad (53)$$

Note that the formulas for the functions $T(\cdots)$'s appearing in Eq. (53) are given by Eqs. (24), (25) and (26). Similarly, the functions $F(\cdots)$'s appearing ahead are given by Eqs. (28) and (37). For the marginal variances, we have

$$\begin{aligned} \widetilde{M}_{20}(m_1, m_2) &= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}})\mathcal{O}(k_{\mathcal{O}})H^2(k_H) \rangle^{m_1, m_2}} \\ &= \widetilde{M}_{00}(m_1, m_2) \overline{\langle H^2(k_H) \rangle^{m_1, m_2}}, \\ \widetilde{M}_{02}(m_1, m_2) &= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}})H^2(k_H)\mathcal{O}(k_{\mathcal{O}}) \rangle^{m_1, m_2}} \\ &= \widetilde{M}_{00}(m_1, m_2) \overline{\langle H^2(k_H) \rangle^{m_1+k_{\mathcal{O}}, m_2-k_{\mathcal{O}}}}. \end{aligned} \quad (54)$$

In Eq. (54), the ensemble averages of $H^2(k_H)$ are given by Eq. (44). Now, the correlation coefficient ζ_{biv} is

$$\zeta_{biv}(m_1, m_2) = \frac{\widetilde{M}_{11}(m_1, m_2)}{\sqrt{\widetilde{M}_{20}(m_1, m_2) \widetilde{M}_{02}(m_1, m_2)}}. \quad (55)$$

Clearly, ζ_{biv} will be independent of $v_{\mathcal{O}}^2$.

Proceeding further, we derive formulas for the fourth order moments \widetilde{M}_{PQ} , $P + Q = 4$. The results are as follows. Firstly, for $(PQ) = (40)$ and (04) , we have

$$\begin{aligned}
\widetilde{M}_{40}(m_1, m_2) &= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \mathcal{O}(k_{\mathcal{O}}) H^4(k_H) \rangle}^{m_1, m_2} \\
&= \widetilde{M}_{00}(m_1, m_2) \overline{\langle H^4(k_H) \rangle}^{m_1, m_2}, \\
\widetilde{M}_{04}(m_1, m_2) &= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) H^4(k_H) \mathcal{O}(k_{\mathcal{O}}) \rangle}^{m_1, m_2} \\
&= \widetilde{M}_{00}(m_1, m_2) \overline{\langle H^4(k_H) \rangle}^{m_1 + k_{\mathcal{O}}, m_2 - k_{\mathcal{O}}}.
\end{aligned} \tag{56}$$

In Eq. (56), the ensemble averages of $H^4(k_H)$ are given by Eq. (46). For $(PQ) = (31)$, we have

$$\begin{aligned}
\widetilde{M}_{31}(m_1, m_2) &= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) H(k_H) \mathcal{O}(k_{\mathcal{O}}) H^3(k_H) \rangle}^{m_1, m_2} \\
&= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \underbrace{H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H)}_{\text{contracted}} \underbrace{H(k_H) H(k_H) H(k_H)}_{\text{contracted}} \rangle}^{m_1, m_2} \\
&+ \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \underbrace{H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H) H(k_H)}_{\text{contracted}} H(k_H) \rangle}^{m_1, m_2} \\
&+ \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \underbrace{H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H) H(k_H)}_{\text{contracted}} H(k_H) \rangle}^{m_1, m_2}.
\end{aligned} \tag{57}$$

First and last terms on RHS of Eq. (57) are simple as \underbrace{HH} can be taken out of the average and then we are left with a term similar to $\widetilde{M}_{11}(m_1, m_2)$. For the second term, the \mathcal{O}^\dagger and \mathcal{O} operators are contracted across H operator using Eqs. (20) and (21) and then one is left with an average of the form $\langle HGHG \rangle$. These will give the final formula,

$$\begin{aligned}
\widetilde{M}_{31}(m_1, m_2) &= 2 \overline{\langle H^2(k_H) \rangle}^{m_1, m_2} \widetilde{M}_{11}(m_1, m_2) \\
&+ \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \underbrace{H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H) H(k_H) H(k_H)}_{\text{contracted}} \rangle}^{m_1, m_2} \\
&= 2 \overline{\langle H^2(k_H) \rangle}^{m_1, m_2} \widetilde{M}_{11}(m_1, m_2) + v_{\mathcal{O}}^2 \sum_{i+j=k_H, t+u=k_H} v_H^2(i, j) v_H^2(t, u) \\
&\times \binom{m_2 - j}{k_{\mathcal{O}}} \binom{\widetilde{m}_1 - i}{k_{\mathcal{O}}} F(m_1, N_1, i, t) F(m_2, N_2, j, u).
\end{aligned} \tag{58}$$

Similarly, we have

$$\begin{aligned}
\widetilde{M}_{13}(m_1, m_2) &= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) H^3(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H) \rangle}^{m_1, m_2} \\
&= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \underbrace{H(k_H) H(k_H)} \underbrace{H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H)} \rangle}^{m_1, m_2} \\
&\quad + \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \underbrace{H(k_H) H(k_H) H(k_H)} \mathcal{O}(k_{\mathcal{O}}) H(k_H) \rangle}^{m_1, m_2} \\
&\quad + \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \underbrace{H(k_H) H(k_H) H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H)} \rangle}^{m_1, m_2} \\
&= 2 \overline{\langle H^2(k_H) \rangle}^{m_1 + k_{\mathcal{O}}, m_2 - k_{\mathcal{O}}} \widetilde{M}_{11}(m_1, m_2)
\end{aligned} \tag{59}$$

$$\begin{aligned}
&+ v_{\mathcal{O}}^2 \sum_{i+j=k_H, t+u=k_H} v_H^2(i, j) v_H^2(t, u) G(t, u) \\
&\times \binom{\widetilde{m}_1 - k_{\mathcal{O}} - t + i}{i} \binom{m_1 + k_{\mathcal{O}} - t}{i} \binom{\widetilde{m}_2 - u + k_{\mathcal{O}} + j}{j} \binom{m_2 - k_{\mathcal{O}} - u}{j} ; \\
&G(t, u) = \binom{\widetilde{m}_1 - t}{k_{\mathcal{O}}} \binom{m_2 - u}{k_{\mathcal{O}}} T(m_1, N_1, t) T(m_2, N_2, u) .
\end{aligned}$$

In Eq. (59), the first and last terms can be evaluated by first calculating the H^2 average over the intermediate states $|m_1 + k_{\mathcal{O}}, m_2 - k_{\mathcal{O}}\rangle$ and then the remaining part is similar to $\widetilde{M}_{11}(m_1, m_2)$. Also, the second average is evaluated by first contracting the two correlated H 's that are between \mathcal{O}^\dagger and \mathcal{O} operators (see the contraction symbol for clarity) and then one is again left with a term similar to $\widetilde{M}_{11}(m_1, m_2)$. Finally, $\widetilde{M}_{22}(m_1, m_2)$ is given by,

$$\begin{aligned}
\widetilde{M}_{22}(m_1, m_2) &= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) H^2(k_H) \mathcal{O}(k_{\mathcal{O}}) H^2(k_H) \rangle}^{m_1, m_2} \\
&= \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \underbrace{H(k_H) H(k_H)} \mathcal{O}(k_{\mathcal{O}}) \underbrace{H(k_H) H(k_H)} \rangle}^{m_1, m_2} \\
&\quad + \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \underbrace{H(k_H) H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H)} \rangle}^{m_1, m_2} \\
&\quad + \overline{\langle \mathcal{O}^\dagger(k_{\mathcal{O}}) \underbrace{H(k_H) H(k_H) \mathcal{O}(k_{\mathcal{O}}) H(k_H) H(k_H)} \rangle}^{m_1, m_2}
\end{aligned}$$

$$\begin{aligned}
&= \widetilde{M}_{00}(m_1, m_2) \overline{\langle H^2(k_H) \rangle^{m_1+k_{\mathcal{O}}, m_2-k_{\mathcal{O}}}} \overline{\langle H^2(k_H) \rangle^{m_1, m_2}} \\
&+ v_{\mathcal{O}}^2 \sum_{i+j=k_H, t+u=k_H} v_H^2(i, j) v_H^2(t, u) \binom{\widetilde{m}_1 - i - t}{k_{\mathcal{O}}} \binom{m_2 - u - j}{k_{\mathcal{O}}} \\
&\times [F(m_1, N_1, i, t) F(m_2, N_2, j, u) \\
&+ T(m_1, N_1, i) T(m_1, N_1, t) T(m_2, N_2, j) T(m_2, N_2, u)] .
\end{aligned} \tag{60}$$

In Eq. (60), the first term is evaluated by first calculating the H^2 average (for the H^2 between \mathcal{O}^\dagger and \mathcal{O} operators) over the intermediate state $|m_1 + k_{\mathcal{O}}, m_2 - k_{\mathcal{O}}\rangle$ and then one is left with product of averages of H^2 and $\mathcal{O}^\dagger \mathcal{O}$ operators. For the third term, first the \mathcal{O}^\dagger and \mathcal{O} operators are contracted across H^2 operator and then we are left with average of the form $\langle H^2 \rangle \times \langle H^2 \rangle$. Similarly, for the second term, after contracting the \mathcal{O}^\dagger and \mathcal{O} operators across H^2 operator, we are left with an average of the form $\langle HGHG \rangle$.

C. Numerical results for bivariate correlation coefficient and fourth order cumulants

Firstly, given the $\widetilde{M}_{PQ}(m_1, m_2)$, the normalized central moments M_{PQ} are $M_{PQ} = \widetilde{M}_{PQ} / \widetilde{M}_{00}$. Then, the scaled moments \widehat{M}_{PQ} are

$$\widehat{M}_{PQ} = \frac{M_{PQ}(m_1, m_2)}{[M_{20}(m_1, m_2)]^{P/2} [M_{02}(m_1, m_2)]^{Q/2}} ; \quad P + Q \geq 2 . \tag{61}$$

Now the fourth order cumulants are [52],

$$\begin{aligned}
k_{40}(m_1, m_2) &= \widehat{M}_{40}(m_1, m_2) - 3 , \quad k_{04}(m_1, m_2) = \widehat{M}_{04}(m_1, m_2) - 3 , \\
k_{31}(m_1, m_2) &= \widehat{M}_{31}(m_1, m_2) - 3 \widehat{M}_{11}(m_1, m_2) , \\
k_{13}(m_1, m_2) &= \widehat{M}_{13}(m_1, m_2) - 3 \widehat{M}_{11}(m_1, m_2) , \\
k_{22}(m_1, m_2) &= \widehat{M}_{22}(m_1, m_2) - 2 \widehat{M}_{11}^2(m_1, m_2) - 1 .
\end{aligned} \tag{62}$$

Assuming $v_H^2(i, j)$ defining $H(2)$ are independent of (i, j) so that ζ_{biv} is independent of v_H^2 , we have calculated the value of ζ_{biv} with $k_{\mathcal{O}} = 2$ for several $0\nu\beta^-\beta^-$ decay candidate nuclei using Eq. (55) along with Eqs. (51), (53), (54) and (44). For the function $T(\cdots)$, we use Eq. (24). Note that $v_H^2(i, j)$ correspond to the variance of two-particle matrix elements from the $p-p$ ($i=2, j=0$), $n-n$ ($i=0, j=2$) and $p-n$ ($i=1, j=1$) interactions. Results are given in Table I. It is seen that $\zeta_{biv} \sim 0.6 - 0.8$. It is important to mention that $\zeta_{biv} = 0$ for GOE. Therefore, the transition

TABLE I. Correlation coefficients $\zeta_{biv}(m_1, m_2)$ for some nuclei with $k_O = 2$ as appropriate for $0\nu\beta^-\beta^-$ decay operator. Note that space #1 is for protons and space #2 for neutrons. The configuration spaces corresponding to N_1 or $N_2 = 20, 22, 30, 32, 44$ and 58 are $r_3f, r_3g, r_4g, r_4h, r_5i$, and r_6j , respectively with $f = {}^1f_{7/2}, g = {}^1g_{9/2}, h = {}^1h_{11/2}, i = {}^1i_{13/2}, j = {}^1j_{15/2}, r_3 = {}^1f_{5/2} {}^2p_{3/2} {}^2p_{1/2}, r_4 = {}^1g_{7/2} {}^2d_{5/2} {}^2d_{3/2} {}^3s_{1/2}, r_5 = {}^1h_{9/2} {}^2f_{7/2} {}^2f_{5/2} {}^3p_{3/2} {}^3p_{1/2}$ and $r_6 = {}^1i_{11/2} {}^2g_{9/2} {}^2g_{7/2} {}^3d_{5/2} {}^3d_{3/2} {}^4s_{1/2}$. See text for details.

Nuclei	N_1	m_1	N_2	m_2	$\zeta_{biv}(m_1, m_2)$
${}^{76}_{32}\text{Ge}_{44}$	22	4	22	16	0.64
${}^{82}_{34}\text{Se}_{48}$	22	6	22	20	0.6
${}^{100}_{42}\text{Mo}_{58}$	30	2	32	8	0.57
${}^{128}_{52}\text{Te}_{76}$	32	2	32	26	0.62
${}^{130}_{52}\text{Te}_{78}$	32	2	32	28	0.58
${}^{150}_{60}\text{Nd}_{90}$	32	10	44	8	0.72
${}^{154}_{62}\text{Sm}_{92}$	32	12	44	10	0.76
${}^{180}_{74}\text{W}_{106}$	32	24	44	24	0.77
${}^{238}_{92}\text{U}_{146}$	44	10	58	20	0.83

strength density will be narrow in (E_i, E_f) plane. In order to establish the bivariate Gaussian form for the $0\nu\beta^-\beta^-$ decay transition strength density, we have examined k_{PQ} , $P + Q = 4$. For a good bivariate Gaussian, $|k_{PQ}| \lesssim 0.3$. Using Eqs. (51), (53), (54), (56), (58)-(62) along with Eqs. (44) and (46), we have calculated the cumulants $k_{PQ}(m_1, m_2)$, $P + Q = 4$. These involve $T(\cdots)$ and $F(\cdots)$ functions. For set #1 calculations in Table II, we use Eq. (24) for $T(\cdots)$ and Eq. (37) for $F(\cdots)$. For the set #2 calculations, shown in ‘brackets’ in Table II, we use Eq. (25) for $T(\cdots)$, Eq. (28) for $F(\cdots)$ and replace everywhere $\binom{\tilde{m}_i+r}{s} \rightarrow \binom{N_i}{s}$ for any (r, s) with $i = 1, 2$. Then we have the strict dilute limit. We show in Table II, bivariate cumulants for five heavy nuclei for both sets of calculations and they clearly establish that bivariate Gaussian is a good approximation (similar tests are made for β decay operator in Appendix A). We have also examined this analytically in the dilute limit with $N_1, N_2 \rightarrow \infty$ and assuming $v_H^2(i, j)$ independent of (i, j) . With these, we have expanded k_{PQ} in powers of $1/m_1$ and $1/m_2$ using Mathematica. It is seen that all the k_{PQ} ,

TABLE II. Cumulants k_{PQ} , $P + Q = 4$ for some nuclei listed in Table I. The numbers in the brackets are for the strict dilute limit as explained in the text. Just as in the construction of Table I, we use $v_H^2(i, j)$ independent of (i, j) . See Table I and text for details.

Nuclei	N_1	m_1	N_2	m_2	k_{40}	k_{04}	k_{13}	k_{31}	k_{22}
$^{100}_{42}\text{Mo}_{58}$	30	2	32	8	$-0.45(-0.39)$	$-0.42(-0.38)$	$-0.24(-0.23)$	$-0.26(-0.25)$	$-0.20(-0.22)$
$^{150}_{60}\text{Nd}_{90}$	32	10	44	8	$-0.27(-0.22)$	$-0.29(-0.23)$	$-0.22(-0.18)$	$-0.20(-0.17)$	$-0.19(-0.18)$
$^{154}_{62}\text{Sm}_{92}$	32	12	44	10	$-0.24(-0.18)$	$-0.25(-0.18)$	$-0.19(-0.15)$	$-0.18(-0.15)$	$-0.17(-0.15)$
$^{180}_{74}\text{W}_{106}$	32	24	44	24	$-0.19(-0.08)$	$-0.20(-0.08)$	$-0.17(-0.08)$	$-0.15(-0.08)$	$-0.15(-0.08)$
$^{238}_{92}\text{U}_{146}$	44	10	58	20	$-0.18(-0.13)$	$-0.18(-0.13)$	$-0.15(-0.11)$	$-0.15(-0.11)$	$-0.13(-0.11)$

$P + Q = 4$ behave as,

$$k_{PQ} = -\frac{4}{m_1} + O\left(\frac{1}{m_1^2}\right) + O\left(\frac{m_2^2}{m_1^3}\right) + \dots \quad (63)$$

Therefore, for $m_1 \gg 1$ and $m_2 \ll m_1^{3/2}$, the strength density approaches bivariate Gaussian form in general. It is important to recall that the strong dependence on m_1 in Eq. (63) is due to the nature of the operator \mathcal{O} i.e., $\mathcal{O}(k_{\mathcal{O}}) |m_1, m_2\rangle = |m_1 + k_{\mathcal{O}}, m_2 - k_{\mathcal{O}}\rangle$. Thus, we conclude that bivariate Gaussian form is a good approximation for $0\nu\beta^-\beta^-$ decay transition strength densities. With this, one can apply the formulation given in Sec. III with the bivariate correlation coefficient ζ_{biv} given by Eqs. (55), (54) and (53). The values given by the two-orbit binary correlation theory for ζ_{biv} can be used as starting values in practical calculations.

VI. CONCLUSIONS

To summarize, by extending the binary correlation approximation method for spinless embedded k -body ensembles to ensembles with proton-neutron degrees of freedom that involves traces involving product of powers of two different operators over two-orbit configurations (either the operators preserve the two-orbit symmetry or change a two-orbit configuration to a unique final configuration), we have established that the transition strength density generated by the two-body part of the Hamiltonian is a bivariate Gaussian for transition operators $\mathcal{O}(k_{\mathcal{O}})$ that change $k_{\mathcal{O}}$ number of neutrons to $k_{\mathcal{O}}$ number of protons. Towards this end, we have derived formulas for the fourth order cumulants of the transition strength density and calculated their values for some

TABLE III. Correlation coefficients $\zeta_{biv}(m_1, m_2)$ and cumulants k_{PQ} , $P + Q = 4$ for some nuclei relevant for β decay. The first four nuclei in the table are relevant for β^- transitions, next four nuclei are relevant for electron capture and the last two nuclei are relevant for β^+ transitions. The numbers in the brackets for k_{PQ} are for the strict dilute limit as in Table II. We assume $v_H^2(i, j)$ are independent of (i, j) as used in the calculations generating Tables I and II. Here, $m_1 = m_p$, $m_2 = m_n$ for the first four nuclei and $m_1 = m_n$, $m_2 = m_p$ for the next six nuclei. See text for details.

Nuclei	N_1	m_1	N_2	m_2	$\zeta_{biv}(m_1, m_2)$	k_{40}	k_{04}	k_{13}	k_{31}	k_{22}
$^{62}_{27}\text{Co}_{35}$	20	7	30	15	0.72	-0.26(-0.18)	-0.27(-0.18)	-0.24(-0.16)	-0.23(-0.16)	-0.22(-0.16)
$^{64}_{27}\text{Co}_{37}$	20	7	30	17	0.73	-0.27(-0.16)	-0.27(-0.16)	-0.24(-0.15)	-0.23(-0.15)	-0.21(-0.15)
$^{62}_{26}\text{Fe}_{36}$	20	6	30	16	0.72	-0.28(-0.18)	-0.28(-0.18)	-0.24(-0.16)	-0.24(-0.16)	-0.22(-0.16)
$^{68}_{28}\text{Ni}_{40}$	20	8	30	20	0.72	-0.27(-0.14)	-0.27(-0.14)	-0.24(-0.13)	-0.23(-0.13)	-0.21(-0.13)
$^{65}_{32}\text{Ge}_{33}$	36	5	36	4	0.55	-0.45(-0.41)	-0.46(-0.42)	-0.35(-0.33)	-0.34(-0.32)	-0.34(-0.34)
$^{69}_{34}\text{Se}_{35}$	36	7	36	6	0.66	-0.36(-0.29)	-0.34(-0.30)	-0.28(-0.25)	-0.28(-0.25)	-0.27(-0.25)
$^{73}_{36}\text{Kr}_{37}$	36	9	36	8	0.72	-0.28(-0.23)	-0.28(-0.23)	-0.24(-0.20)	-0.24(-0.20)	-0.23(-0.20)
$^{77}_{38}\text{Sr}_{39}$	36	11	36	10	0.76	-0.24(-0.19)	-0.24(-0.19)	-0.21(-0.17)	-0.21(-0.17)	-0.20(-0.17)
$^{85}_{42}\text{Mo}_{43}$	36	15	36	14	0.79	-0.20(-0.14)	-0.21(-0.14)	-0.19(-0.13)	-0.18(-0.13)	-0.17(-0.13)
$^{93}_{46}\text{Pd}_{47}$	36	19	36	18	0.80	-0.19(-0.11)	-0.19(-0.11)	-0.18(-0.10)	-0.17(-0.10)	-0.16(-0.10)

realistic examples; they are found to vary from ~ -0.4 to -0.1 . It is important to mention that the embedding algebra for the EGOEs used is $U(N) \supset U(N_p) \oplus U(N_n)$ [p denotes ‘protons’ and n denotes ‘neutrons’] with the Hamiltonian preserving the symmetry and the transition operator breaking the symmetry in a particular way. We have also derived a formula for the fourth order trace defining the correlation coefficient of the bivariate transition strength density for the transition operator relevant for $0\nu\beta^-\beta^-$ decay. For nuclei from ^{76}Ge to ^{238}U , the bivariate correlation coefficient is found to vary from $\sim 0.6 - 0.8$ and these values can be used as a starting point for calculating nuclear transition matrix elements for NDBD using the spectral distribution method outlined in Sec. III. In future, it is important to test the approximations leading to Eq. (14) using shell model examples. Although spectral distribution method is expected to be valid in the chaotic domain of the spectrum (usually away from the ground state), it remains to be tested how well the method applies to the calculation of NTME for NDBD. In the past, the theory has been applied successfully for occupancies near the ground state [19, 53–55] and also it is shown that in the level density analysis of heavy nuclei [44] that the theory extends close to the ground state. In the near future, applications will be carried out for NTME for some heavy nuclei (^{100}Mo , ^{154}Sm , ^{150}Nd , ^{186}W , ^{238}U).

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APPENDIX A

For completeness, we have also calculated the correlation coefficient and fourth order moments for the transition operator relevant for β decay [$k_{\mathcal{O}} = 1$ in Eq. (47)]. Results are given in Table III. For the first four nuclei (they are β^- candidates) in the table, $N = Z = 20$ is used as the core. Here, N_1 corresponds to $^1f_{7/2} \ ^1f_{5/2} \ ^2p_{3/2} \ ^2p_{1/2}$ and N_2 corresponds to $^1f_{7/2} \ ^1f_{5/2} \ ^2p_{3/2} \ ^2p_{1/2} \ ^1g_{9/2}$. Similarly, for the remaining six nuclei (they are electron capture or β^+ candidates), $N = Z = 28$ and N_1 and N_2 correspond to $^1f_{5/2} \ ^2p_{3/2} \ ^2p_{1/2} \ ^1g_{9/2} \ ^1g_{7/2} \ ^2d_{5/2}$. The fourth order cumulants values presented in Table III confirm that the bivariate Gaussian form is a good approximation for β decay transition strength densities. Results in Table III justify the assumptions made in [39, 56] where

spectral distribution method is applied, with the correlation coefficients in the correct range, to calculate the β decay rates for nuclei relevant for pre-supernovae evolution.

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